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A MECHANICAL MODEL OF THE MOTION OF A LIMITED VOLUME OF LIQUID ON A DRY INCLINED PLANE

M. I. Abu-Halava and M. T. Gladyshev UDC 532.5:627

We investigate a new mechanical model of the motion of an outburst wave on a dry slope.

An overview of the works done by one of the present authors on an outburst wave is given in [1]. One of the models for the motion of a limited volume of liquid on a dry river bed is proposed in [2, 3]. In particular, St. Venant's model, while taking friction into account, "freezes" the trailing edge [1] in contradiction with the physics of the phenomenon. In contrast with the approach based on the solution of the St. Venant equations, we consider a model that makes it possible to take into consideration some factors that are not taken into account in the St. Venant model.

1. Description of the Model. A limited volume of incompressible homogeneous liquid is characterized in the model by two parameters that specify the shape of the volume. Motion is described by the velocity with which its center of gravity moves and by the change in the parameters of the shape. In this case the following factors are taken into account: the action of the gravitational force, relative motion within the outburst wave, turbulent friction against the slope, the lift force, and the resistance of the surrounding air. In composing the system of equations it is assumed for simplicity that the slope has a constant angle of inclination to the horizon in the direction across the motion. This allows one to neglect the lateral velocity and to consider the phenomenon to be plane. The limited volume of the liquid is considered per unit length of its front and is quantitatively equal to the area of the cross section perpendicular to the front. This cross section is assumed to have the form of an ellipse that is somewhat deformed in motion. This makes it possible to prescribe the shape and dimensions of the outburst wave by just two parameters, for example, the lengths of the semiaxes of the ellipse or its height and area. We will take the latter for what follows.

Thus the motion of a limited volume of liquid is represented not by a model of a continuous medium, but by a mechanical model with a finite number of degrees of freedom. At the same time this motion is described not by the displacement of a material point, but by adding the degrees of freedom to describe the deformation of the outburst wave.

For the change in the height of the outburst wave and the velocity of its center of gravity we propose the following system of ordinary differential equations:

$$\frac{dM}{dt} = 0\,, \tag{1}$$

$$\frac{d\,(Mv)}{dt} = Mg\sin\alpha - 2\rho klv^2 - \frac{1}{2}c_*\rho_a hv^2\,, \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot h}\right) - \frac{\partial T}{\partial h} = Q\,. \tag{3}$$

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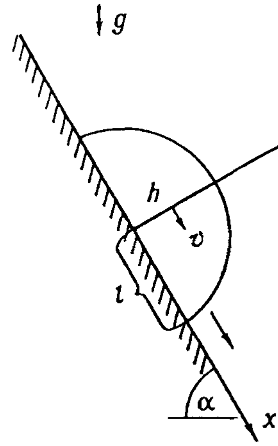


Fig. 1. Scheme of the disposition of the limited volume of liquid on the slope.

The scheme of the disposition of the limited volume of liquid on the slope is given in Fig. 1.

Equation (1) gives the mass-conservation law of the outburst wave, i.e., S = const at ρ = const; Eq. (2) shows its momentum Mv acquired under the effect of gravity, friction against the slope, and air resistance, while Eq. (3) is the Lagrange equation for motion relative to the center of gravity.

We shall give the specific form of the formulas adopted to calculate the terms T and Q. To calculate the kinetic energy T of internal motion, the following law of the distribution of velocities within the limited volume of liquid was adopted: $V_x = (x/l)\dot{l}$, $V_y = (y/h)\dot{h}$, which ensures that the following conditions will be satisfied on the boundary: $V_x = (x = \pm l) = \pm\dot{l}$, $V_y(y = h) = \dot{h}$. With the law of motion taken in this form, the kinetic energy of the deformation of the outburst wave turns out to be equal to $T = (1/8)M(\dot{h}^2 + \dot{l}^2)$. Considering that l and h are interrelated as $S = \pi lh/2$, we finally obtain

$$T = \frac{1}{8} M \left(1 + \frac{4S^2}{\pi^2 h^4}\right) \dot{h}^2 . \tag{4}$$

The generalized force Q is calculated as the work ΔA of all the external forces done to deform the ellipse Δh, $Q = \Delta A/\Delta h$.

The acting forces are the force of gravity and the distribution of pressure on the free boundary of the limited volume of liquid. The gravitational force tends to displace its center of gravity downward, i.e., to flatten the ellipse. The distribution of pressure over the surface of the ellipse in the counterflow of air is such that it tends to extend the upper boundary of the limited volume of liquid upward (lift force). Thus these forces act in opposition. Their resultant value turns out to be equal to

$$Q = -\frac{4}{3\pi}\left(1 - \frac{\rho_a}{\rho}\right) Mg\cos\alpha + \frac{\pi}{2}\rho_a h v^2 . \tag{5}$$

Since the flow past the outburst wave will not be separationless, as was assumed when the latter formula was derived, we introduce a certain coefficient, taken equal to 0.7, into the expression for the lift force. Moreover, in the first term in (5) we neglect the ratio ρ_a/ρ compared to 1, since the densities ρ and ρ_a are constant and the value of ρ_a/ρ is small.

Now, taking into account these remarks, we substitute (4) and (5) into system (1)-(3). We write it in terms of dimensionless variables. For this purpose, we select a certain scale L to determine length; then, it is natural to take $\sqrt{Lg}$ to measure velocity and $\sqrt{L/g}$ to find the time. As the scale for measuring density we will take a certain ρ_*, for example, the density of air. To denote the dimensionless values of the parameters, we use the same symbols with a bar. For convenience S will be replaced by $\bar{\bar{S}} = \bar{S}/\pi = \bar{h}\,\bar{l}$, and then $\bar{l} = \bar{\bar{S}}/\bar{h}$. Finally, we denote $dh/dt = w$. This is done for all the equations to be of first order in the system. Thus, we have the formulas

$$\bar{t} = t\sqrt{\left(\frac{g}{L}\right)}, \quad \bar{h} = \frac{h}{L}, \quad \bar{v} = \frac{v}{\sqrt{Lg}}, \quad \bar{w} = \frac{w}{\sqrt{Lg}}, \quad \bar{S} = \frac{2S}{\pi L^2}, \quad \bar{\rho} = \frac{\rho}{\rho_*}.$$

For the quantities with a bar, which is dropped, the following system is obtained:

$$\frac{dv}{dt} = \sin\alpha - \frac{4k}{\pi}\frac{v^2}{h} - \frac{c_*}{S}\frac{\rho_a}{\rho}hv^2, \quad \frac{dh}{dt} = w,$$
$$\frac{dw}{dt} = \frac{4}{1 + \frac{S^2}{h^4}}\left(\frac{1}{2}\frac{S^2}{h^5}w^2 - \frac{4}{3\pi}\cos\alpha + 0.7\frac{\pi\rho_a}{S\rho}hv^2\right). \tag{6}$$

System (6) was investigated analytically and numerically on a computer.

2. Stationary Solution. It is seen that system (6) has a stationary singular point, i.e., the right-hand sides of all three equations of system (6) can vanish simultaneously. Physically this means that there is motion of a limited volume of liquid with the center of gravity moving with constant velocity and with the dimensions nonvariable. The parameters of this stationary motion obey the following requirements:

$$w = 0, \quad \sin\alpha - \frac{4k}{\pi}\frac{v^2}{h} - \frac{c_*\rho_a}{S\rho}hv^2 = 0, \quad \frac{4}{3\pi}\cos\alpha - 0.7\frac{\pi\rho_a}{S\rho}hv^2 = 0. \tag{7}$$

When Eqs. (7) are satisfied, stationary motion occurs. The stationary values of h and v have the form

$$h_*^2 = \frac{\frac{16}{3\pi^2}k\cos\alpha\frac{S\rho}{\rho_a}}{0.7\sin\alpha - \frac{4}{3\pi^2}c_*\cos\alpha}, \quad v_*^4 = \frac{\cos\alpha}{3\pi}\frac{\rho S}{\rho_a}\frac{0.7\sin\alpha - \frac{4}{3\pi^2}c_*\cos\alpha}{0.49k}.$$

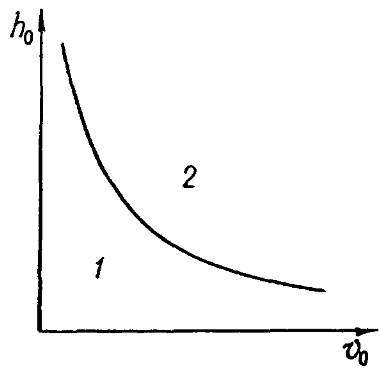
It is seen that S and ρ/ρ_a can take any values here, it is only necessary that the inequality $2.1\pi^2$ tanh $\alpha - 4c_* > 0$ be valid. A denser liquid of the same limited volume has a higher stationary height. The effect of the other parameters is similar. Thus, in the three-dimensional phase space h, v, w the singular stationary point has the coordinates $h = h_*$, $v = v_*$, $w = 0$. Investigation of the approximate form of the integral curves near this singular point shows that it is a focus-saddle. There is a separating surface in h, v, w space that consists of integral curves entering the singular point. The states that are represented by the points of this surface tend to stationary motion. With any small deviation of the initial state h_0, v_0, w_0 from this surface the integral curve withdraws from the singular point. Thus, the stationary point is unstable.

3. Nonstationary Motion of the Outburst Wave. The space of the initial data h_0, v_0, w_0 is divided into two regions by the surface of stationary solutions. On all the integral curves that have their origin in one region there is a rapid increase in the value of h. On the integral curves that originate in the second region, h tends to zero, indicating flattening of the limited volume of liquid and a decrease in the rate of displacement of its center of gravity.

In computer calculations of system (6) the following parameters were taken: $k = 0.02$, $\rho/\rho_a = 20$, $S = 10$.

For a constant angle of the slope of 37 and 20° the curve of the intersection of the separating surface with the plane $w = 0$ was found. The states leading to an increase in the height of the limited volume of liquid lie above the indicated curves, and those leading to its flattening lie below these curves.

Calculation of the motion of the outburst wave on a slope of constant or piecewise-constant steepness from the initial state h_0, v_0, $w_0 = 0$ and its deceleration on a horizontal portion was performed. The main difficulty resides in the selection of coefficients that show their effect and the effect of the initial data on the behavior of the integral



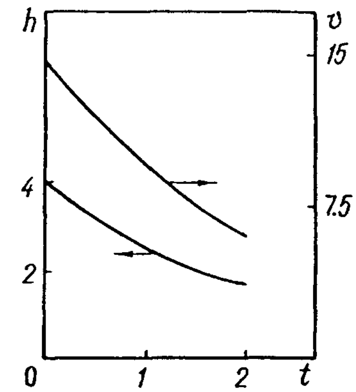


Fig. 2. Division of the plane of initial data into two regions: 1) flattening of the limited volume of liquid, 2) increase of its height.

Fig. 3. Deceleration of the limited volume of liquid on a horizontal portion.

curves. The initial parameters in dimensionless variables were varied within the ranges $h_0 = 1-15$, $v_0 = 0-13$, $w_0 = 0$. The angle of the slope is 37°. The aerodynamic-drag factor is taken to be equal to 1.

The following facts were noted in the investigated ranges of the indicated quantities. Depending on the initial values h_0 and v_0 (see Fig. 2) the integral curves fall into two groups. An increase in the value of h is observed for a curve of the first group. The length of the outburst wave decreases in this case; the wave extends upward and hardly retains its elliptic shape in real motion. Therefore, it would be unreasonable to use the suggested equations further. Other models are needed to study such waves.

In the second group of integral curves the value of h decreases. The length of the outburst wave increases in this case.

In both cases the rate of displacement of the leading front of the limited volume of liquid, which differs from v by the quantity $\dot{l}$ because of its spreading over the slope, was calculated. It is equal to $v - (S/h^2)\dot{h}$.

4. Motion of the Outburst Wave on a Horizontal Plane. The behavior of all the parameters of the limited volume of liquid was calculated for the case where the volume reached a horizontal portion of the slope after motion down an inclined plane. For this purpose the results of the previous calculations at $\alpha = 20°$ were taken as the initial data at $t = 5$. In all the calculations the rate of motion of the center of gravity of the outburst wave v decreases with time, while its height h both decreases and increases with time. Therefore the velocity of the leading edge $v + \dot{l}$ both exceeds v and is less than v. The integral curves $h(t)$ and $v(t)$ are depicted in Fig. 3 for $0 \le t \le 2$.

We note in conclusion that the present approach can be extended to the motion of a limited volume of a water–air mixture when either inflow or outflow of air occurs and to the two-dimensional problem if the limited volume of liquid is an ellipsoid. It is possible to create a model of the motion of an outburst wave under conditions where the slope, on which it propagates, itself moves with a prescribed velocity. In this case the equations are supplemented with a number of terms in connection with conversion to a moving coordinate system.

NOTATION

M, S, ρ, mass, area, and density of the limited volume of liquid; v, rate of displacement of the center of gravity; h, l, linear dimensions of the ellipse (h is the semiaxis normal to the slope, l is the semiaxis along the slope); ρ_a, density of the surrounding air; k, coefficient of turbulent friction against the slope; c_*, aerodynamic-drag factor; g, free-fall acceleration; α, angle of the slope; T, kinetic energy of the relative motion of the liquid in the interior of the limited volume; Q, resultant force; $\dot{h}$, $\dot{l}$, rates of deformation of the boundary of the ellipse at the corresponding points; ΔA, work done by all external forces to deform the ellipse; L, scale for measuring length; $w = dh/dt$; t, time; ρ_*, scale for measuring density; x, coordinate along the slope; y, coordinate perpendicular to the slope; V_x, V_y, components of the velocity of the liquid along the x and y axes; bar, dimensionless values of the parameters; two bars, normalization of the area.

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